—Chapter 5—

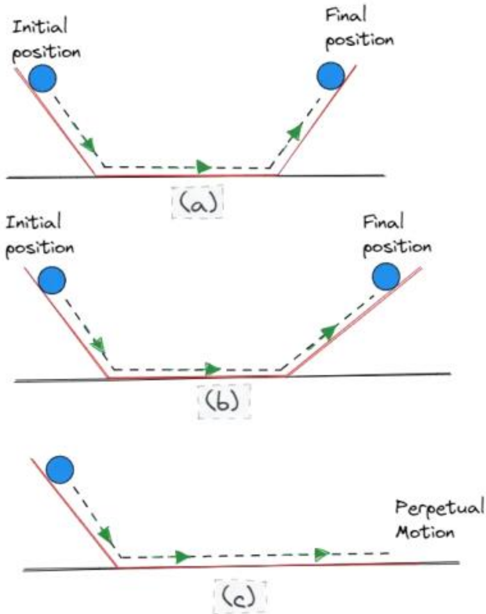
Special Relativity and The Fields of Moving Charges

5-1 Einstein's Postulations in Special Relativity

A. PRIOR TO 1905

I. Inertia Frame of Reference

(1) Galileo's experiments on two incline planes combined together



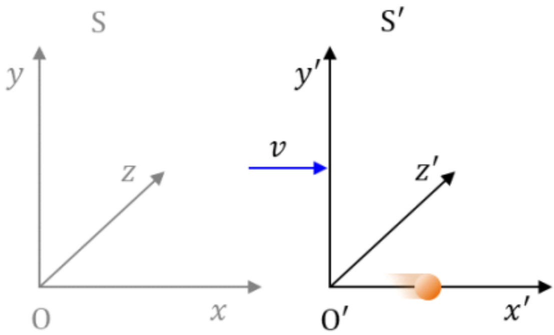
Galileo did the phenomenological description of the concept of inertia:

> An object in a state of motion possesses an "inertia" that causes it to remain in that state of motion unless an external force acts on it.

(2) Newton did the quantitaive description of the concept of inertia and formulated the first law of motion:

> A body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force.

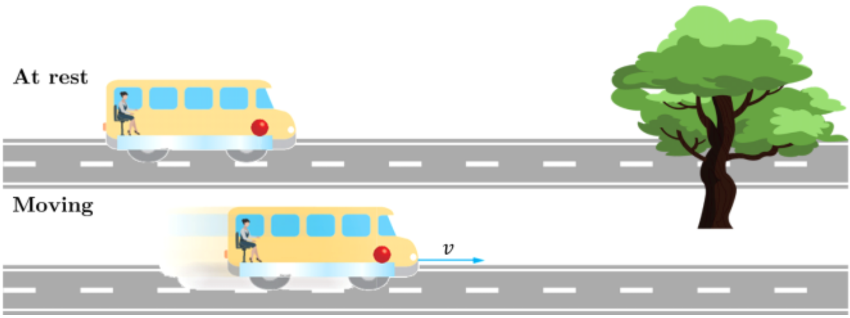
There exists frames of reference such that the velocity of a frame of a moving body is constant or a given frame is at rest in the absence of the external force, then these frames are called inertial frames of reference.



That is, a frame of reference moving with constant velocity relative to an inertial frame is also inertial. A frame of reference accelerating relative to an inertial frame is not inertial.

EXAMPLES:

1. An observer sitting inside a bus does not experience an external force.



Case I:

The ball and the tree are at rest relative to the observer.

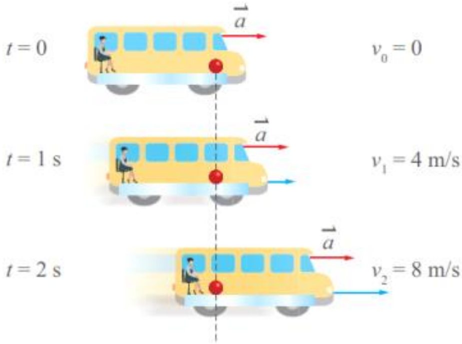
Case II:

The ball is at rest relative to the observer. The tree is moving uniformly toward the observer.

Conclusion:

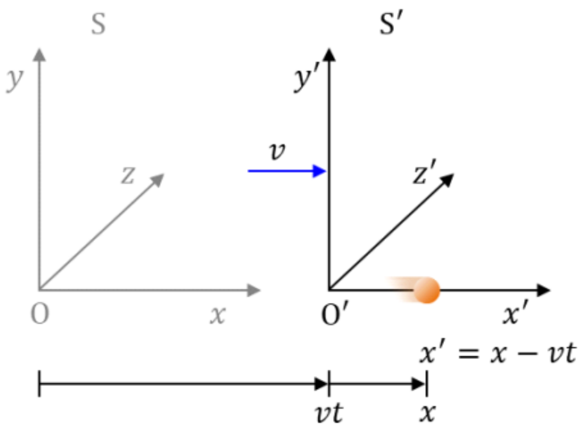
Frames of reference at rest or moving uniformly relative to the observer are called inertial frames of reference.

2. When the bus is accelerating with a,



The ball is accelerating toward the observer while there is no external force acting on it, i.e., the frame of reference of the ball is accelerating relative to the observer called the non-inertial frame of reference.

(3) Consider two inertial frames. The frame S′ is moving forward with constant velocity v relative to the frame S.



The relationships between two inertial frames can be described as:

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \cdots\cdots \text{Galilean transformation}$$

(4) Consequences of Galilean transformation

1. Addition of velocity:

$$\frac{dx'}{dt'} = \frac{d}{dt}(x - vt) = \frac{dx}{dt} - v \Rightarrow u_x' = u_x - v$$

2. Invariant in form of Newton's equation:

PROOF:

$$F' = \frac{dp'}{dt'} = m\frac{du'}{dt'}$$

Since $u' = u - v$

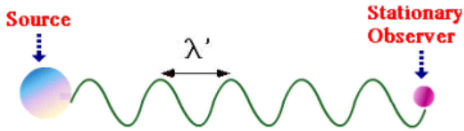
$$\Rightarrow \frac{du'}{dt'} = \frac{d}{dt}(u - v) = \frac{du}{dt}$$

$$\Rightarrow F' = m\frac{du'}{dt'} = m\frac{du}{dt} = F$$

■

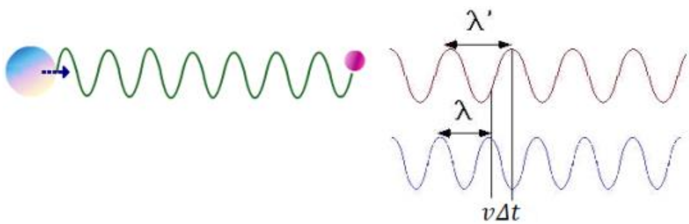
EXAMPLES:

1. Longitudinal Doppler effect



$$f_0 = \frac{u}{\lambda'} = \frac{N}{\Delta t'}$$

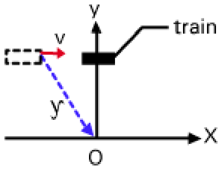
Considering the source approaches observer

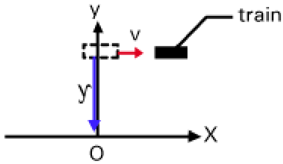


$$\frac{u\Delta t - v\Delta t}{N} = \lambda$$

$$f = \frac{u}{\lambda} = \frac{u}{(u - v)\Delta t/N} = \frac{u}{u - v}\frac{N}{\Delta t} = \frac{u}{u - v}\frac{N}{\Delta t'} = \frac{u}{u - v}f_0$$

2. Transverse Doppler effect

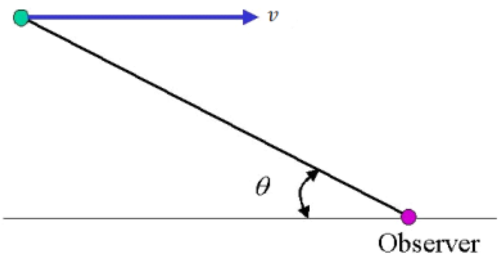




$$f = \frac{N}{\Delta t} = \frac{N}{\Delta t'} = f_0$$

⇒ Doppler effect provides a direct evidence to exhibit the validation of the Galilean transformation.

3. EM Radiation Source

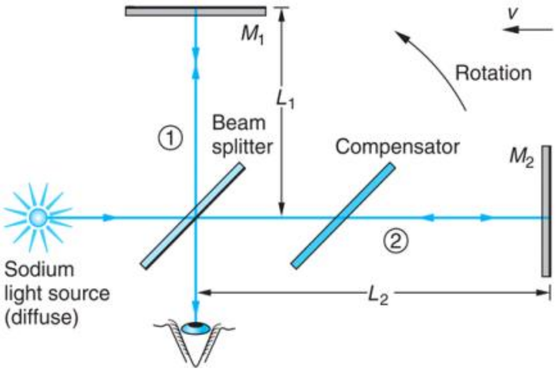


$$f = \frac{u}{u - v\cos\theta} f_0$$

Ⅱ. The Michelson-Morley Experiment

(1) Michelson interferometer and ether

Setup:



Observe the shift in the interference pattern

v along L_2

$$t_2 = \frac{L_2}{c-v} + \frac{L_2}{c+v} = \frac{2L_2}{c}\frac{1}{1-v^2/c^2}$$

$$t_1 = \frac{2L_1}{\sqrt{c^2-v^2}} = \frac{2L_1}{c}\frac{1}{\sqrt{1-v^2/c^2}}$$

$$\Delta t = t_2 - t_1 = \frac{2L_2}{c}\frac{1}{1-v^2/c^2} - \frac{2L_1}{c}\frac{1}{\sqrt{1-v^2/c^2}}$$

v along L_1(rotate 90^0)

$$\Delta t' = t_2' - t_1' = \frac{2L_2}{c}\frac{1}{\sqrt{1-v^2/c^2}} - \frac{2L_1}{c}\frac{1}{1-v^2/c^2}$$

Path difference $\delta x = c\delta t$

$$\delta t = \Delta t - \Delta t' = \frac{2(L_1+L_2)}{c}\left[\frac{1}{1-v^2/c^2} - \frac{1}{\sqrt{1-v^2/c^2}}\right]$$

As $v \ll c$

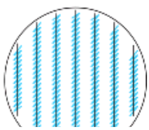
$$\delta t \approx \frac{2(L_1+L_2)}{c}\left[\left(1+\frac{v^2}{c^2}+\cdots\right) - \left(1+\frac{v^2}{2c^2}+\cdots\right)\right] = \frac{v^2}{c^3}(L_1+L_2)$$

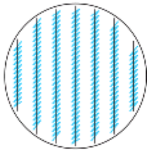
Phase difference $\Delta\phi$

$$\Delta\phi = 2\pi\frac{\delta x}{\lambda} = 2\pi\frac{c\delta t}{\lambda} = \frac{2\pi v^2}{\lambda c^3}(L_1+L_2)$$

EXAMPLES:

1. $v = 3\times 10^4$ m/s, $L_1 = L_2 = 1.2$ m, $\lambda = 600$ nm, $\Delta\phi = 0.02\pi$
   Width of fringe $d = \delta x/\lambda = 0.01$ is detectable



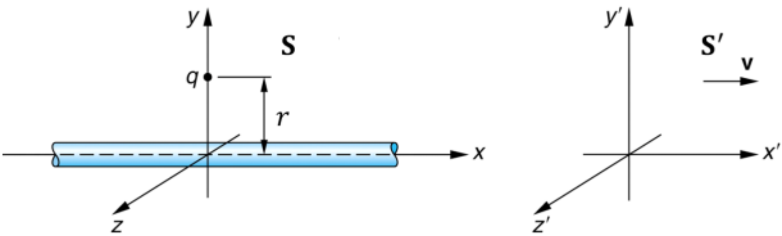


(2) Experimental results:
The interference pattern was not changed after rotation.
Conclusion: The speed of light is the same in all inertial frames.

B. EINSTEIN'S POSTULATIONS

(1) The law of physics is the same in all inertial frames of reference.
COMMENTS:

1. Equations of motion are covariant, invariant in form, under a transformation between two inertial frames, i.e.,
$$\vec{F} = \frac{d\vec{p}}{dt}$$
and
$$\vec{F}' = \frac{d\vec{p}'}{dt}$$
in another frame of reference.
2. Galilean transformation is failed to preserve the form of Maxwell's equations between two inertial frames.
PROOF:
Consider an infinity long wire carrying uniform charge λ and a test charge $+q$ at r.



In S frame:
$$\oint \vec{E} \cdot d\vec{a} = E \cdot 2\pi r l = \frac{Q}{\epsilon_0} \Rightarrow \vec{E} = \frac{Q}{2\pi\epsilon_0 r l} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$
$$\vec{B} = 0$$

$$\vec{F}_y = q\vec{E} + q\vec{v} \times \underbrace{\vec{B}}_{=0} = \frac{q\lambda}{2\pi\epsilon_0 r}\hat{y}$$

In S' frame:

$$\vec{E}' = \vec{E} = \frac{\lambda'}{2\pi\epsilon_0 r'}\hat{r}'$$

$$\oint \vec{B}' \cdot d\vec{l}' = B' \cdot 2\pi r' = \mu_0 I = \mu_0\lambda' v \Rightarrow \vec{B}' = -\frac{\mu_0\lambda' v}{2\pi r'}\hat{z}'$$

$$\vec{F}_y' = q\vec{E}_y' + qvB'\hat{y}' = \frac{q\lambda'}{2\pi\epsilon_0 r}\hat{y}' - \frac{\mu_0 q\lambda' v^2}{2\pi r}\hat{y}'$$

Since

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

we obtain

$$\vec{F}_y' = \frac{q\lambda'}{2\pi\epsilon_0 r}\hat{y}' - \frac{q\lambda' v^2}{2\pi\epsilon_0 r c^2}\hat{y}' = \frac{q\lambda'}{2\pi\epsilon_0 r}\left(1 - \frac{v^2}{c^2}\right)\hat{y}'$$

$$\Rightarrow \vec{F}_y \neq \vec{F}_y'$$

■

(2) The speed of light is equal to the value c (in vacuum) independent of the motion of the light source.

COMMENTS:

1. The speed of light is constant in all inertial frames.
2. The addition of velocity under Galilean transformation is contradictory to Michelson-Morley's experimental results.

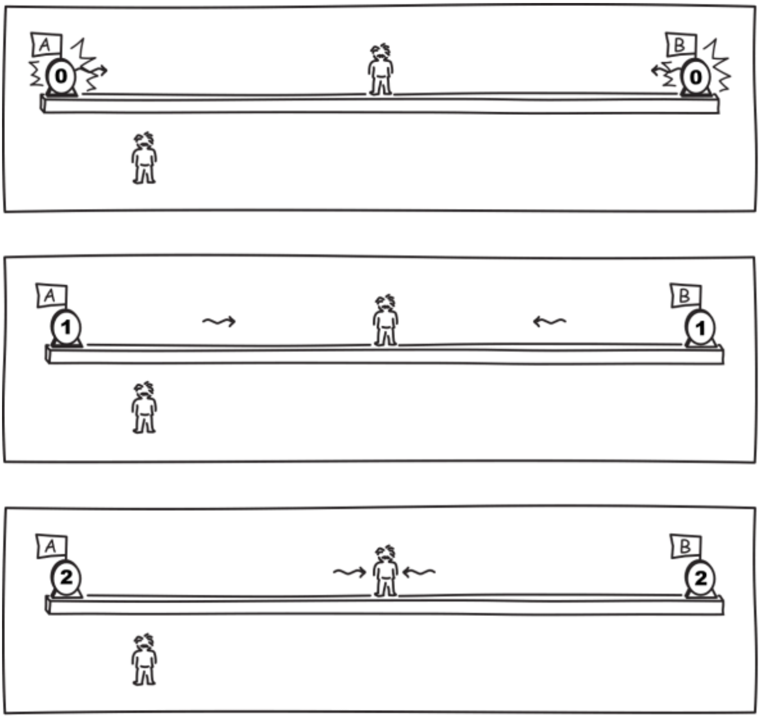
C. CONSEQUENCES OF EINSTEIN'S POSTULATIONS

(1) Relativity of simultaneity

OS:

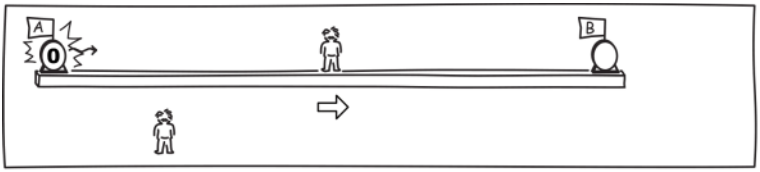
Einstein for Everyone by John D. Norton
https://sites.pitt.edu/~jdnorton/teaching/HPS_0410/index.html

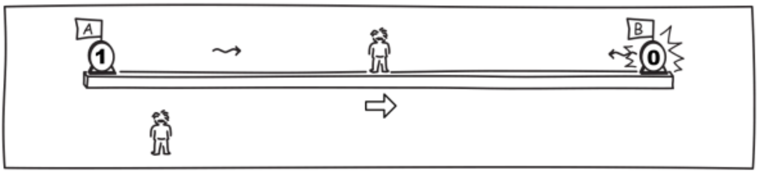
Imagine a long platform with an observer located at its midpoint. At either end, at the places marked A and B, there are two momentary flashes of light.

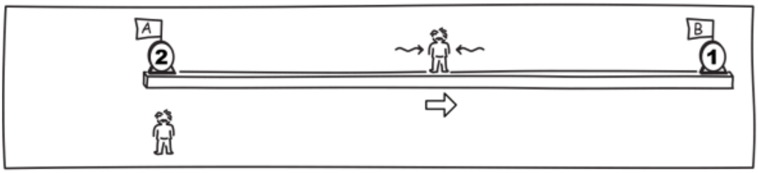


The platform observer judges the A- and B-events to be simultaneous and the A-clock and B-clock to be properly synchronized.

Then consider an observer who moves relative to the platform along its length.





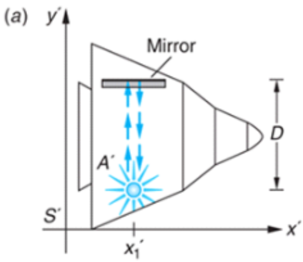


The platform observer judges that the A-event occurs earlier and B-event occurs later. The reasoning extends to the clocks. The A-clock is set earlier than the B-clock, i.e., the clocks at A and B are NOT properly synchronized.

Two events that are simultaneous in one inertial frame of reference will not necessarily be simultaneous in any other inertial frame of reference.

(2) Time dilation

Observer A′ and the light source are at rest in the spaceship.

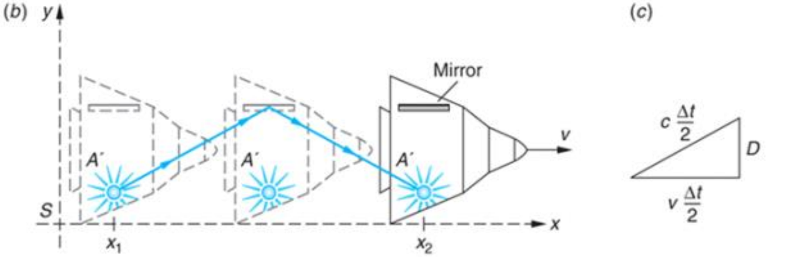


Thus, relative to the light source, the observer A′ is in a rest frame (S′).

From A′:

$$D = c\frac{\Delta t'}{2}$$

Observer A is at rest on the ground.



The observer A, relative to the light source, is in a moving frame (S).
From A:

$$\left(c\frac{\Delta t}{2}\right)^2=\left(v\frac{\Delta t}{2}\right)^2+D^2$$

$$(c^2-v^2)\frac{(\Delta t)^2}{4}=c^2\frac{(\Delta t')^2}{4}$$

$$(\Delta t)^2=\frac{c^2}{c^2-v^2}(\Delta t')^2$$

$$\Delta t=\frac{c\Delta t'}{\sqrt{c^2-v^2}}=\frac{\Delta t'}{\sqrt{1-v^2/c^2}}$$

Let $\gamma=\dfrac{1}{\sqrt{1-v^2/c^2}}>1$

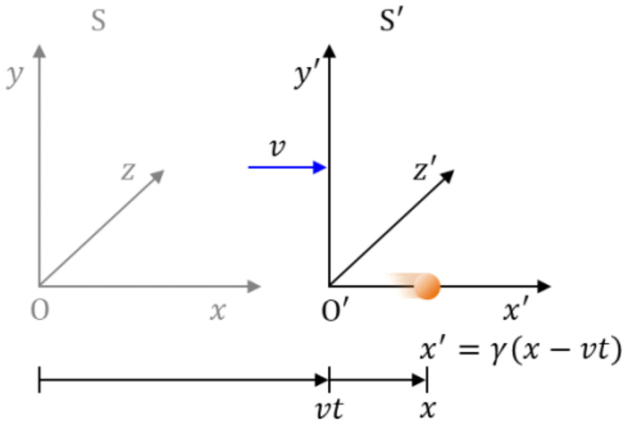
$\Rightarrow \Delta t=\gamma\Delta t'$

where $\Delta t'\equiv\Delta\tau$ is called proper time, which is measured by A′, i.e., during the measurement, A′ is at rest with respect to the event.

5-2 Lorentz Transformation

A. LORENTZ TRANSFORMATION

(1) Lorentz transformation



$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t + \delta)$$

The path of light in S frame

$$x^2 + y^2 + z^2 = c^2t^2 \Rightarrow c^2t^2 - x^2 = y^2 + z^2$$

The path of light in S′ frame

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \Rightarrow c^2t'^2 - x'^2 = y'^2 + z'^2$$

Since

$$y^2 + z^2 = y'^2 + z'^2$$

we obtain

$$\begin{aligned} c^2t^2 - x^2 &= c^2t'^2 - x'^2 \\ &= c^2\gamma^2(t+\delta)^2 - \gamma^2(x - vt)^2 \\ &= c^2\gamma^2\left(t^2 + \delta^2 + 2t\delta\right) - \gamma^2\left(x^2 + v^2t^2 - 2xvt\right) \\ &= \left(c^2\gamma^2 - v^2\gamma^2\right)t^2 + c^2\gamma^2\delta^2 - \gamma^2x^2 + \underbrace{c^2\gamma^2 2t\delta - \gamma^2 2xvt} \end{aligned}$$

Since there is no x and t terms on the L.H.S., we should not have x and t terms on the R.H.S. Thus, we set

$$c^2\gamma^2 2t\delta - \gamma^2 2xvt = 0 \Rightarrow \delta = -\frac{v}{c^2}x$$

$$c^2t^2 - x^2 = (c^2\gamma^2 - v^2\gamma^2)t^2 + c^2\gamma^2\left(\frac{v}{c^2}x\right)^2 - \gamma^2x^2$$

$$= (c^2\gamma^2 - v^2\gamma^2)t^2 + \left(\frac{v^2\gamma^2}{c^2} - \gamma^2\right)x^2$$

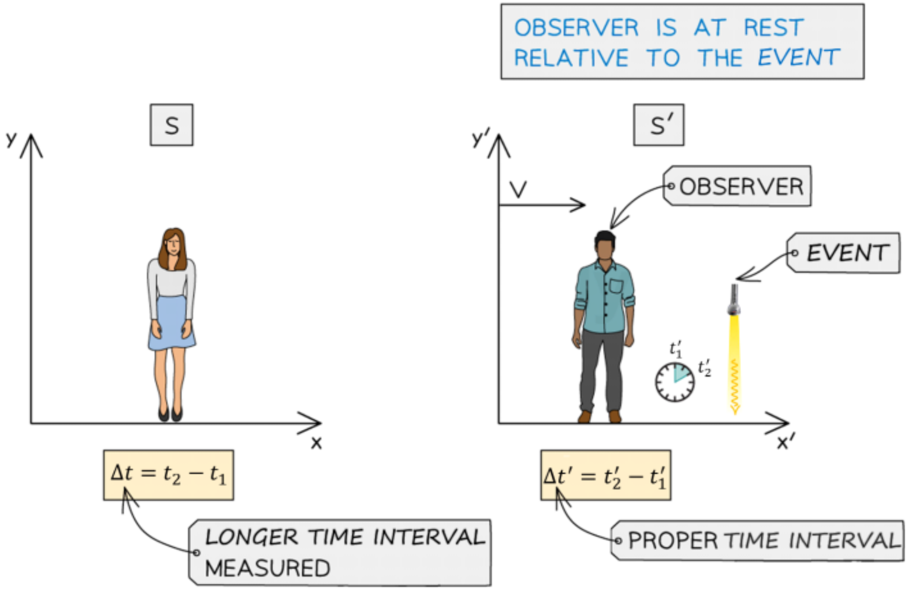
Compare t^2 term (or compare x^2 term, we should obtain the same result):

$$c^2 = c^2\gamma^2 - v^2\gamma^2 \Rightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1$$

Therefore, we obtain the Lorentz transformation

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}, \text{ where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

(2) The time interval measured in S′ frame where the event is at rest relative to the observer.



$$\Delta t' = t_2' - t_1' \equiv \Delta\tau \text{ (proper time in S}' \text{ frame)}$$

From inverse Lorentz transformation

$$t_1 = \gamma\left(t_1' + \frac{v}{c^2}x_1'\right)$$

$$t_2 = \gamma\left(t_2' + \frac{v}{c^2}x_2'\right)$$

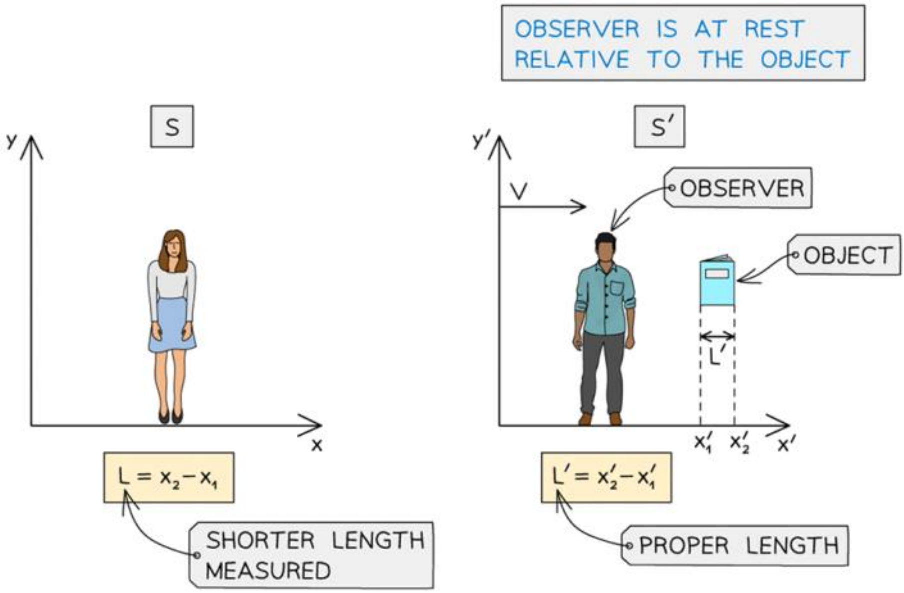
$$\Delta t = t_2 - t_1 = \gamma \Delta t' + \frac{\gamma v}{c^2} \Delta x'$$

Since $\Delta x' = 0$

$$\Delta t = \gamma \Delta t'$$

Since $\gamma > 1$, the observer in S frame measures a longer time interval than the observer in S′ frame.

(3) The length measured in S′ frame where the object is at rest relative to the observer.



$L' = x_2' - x_1'$ (proper length in S′ frame)

From Lorentz transformation

$$x_1' = \gamma(x_1 - vt_1)$$

$$x_2' = \gamma(x_2 - vt_2)$$

$$L' = \gamma L - \gamma v \Delta t$$

Since $\Delta t = 0$

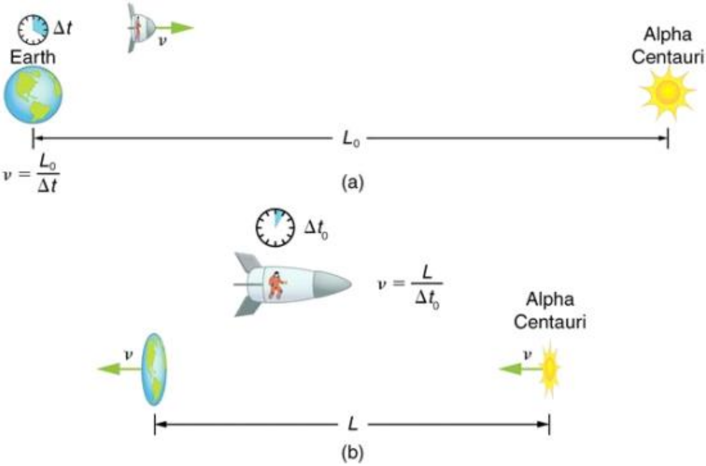
$$L = \frac{L'}{\gamma}$$

Since $\gamma > 1$, the observer in S frame measures a shorter length than the observer in S′ frame.

EXAMPLES:

1. The rocket moves away from us so it becomes shorter for us. In

the reference frame of the astronaut we are the ones moving so we are contracted.



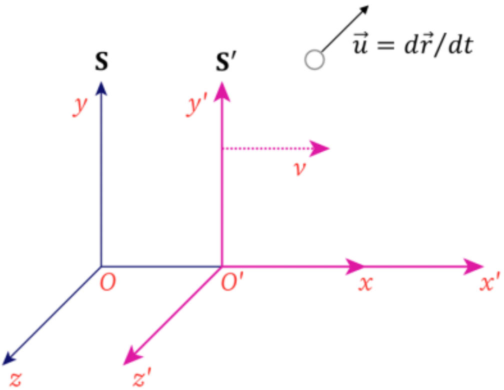
2. Muon decay:
   The speed of μ particle in the cosmic ray is $v = 0.998c$
   The decay time (proper time) $\Delta t' = \Delta\tau = 2\ \mu s$
   The travel distance (proper length) $L' = v\Delta\tau = 600$ m
   Usually, muon comes from at $h = 9000$ m above the sea level.
   Why we can still detect it?
   The decay time at earth is

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} = 30\ \mu\text{s}$$

$$h = 0.998c \times 30 = 9000\ \text{m}$$

B. RELATIVISTIC ADDITION OF VELOCITIES

(1) Relativistic addition of velocities



$$u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \frac{dx'}{dt}\Big/\frac{dt'}{dt}$$
$$u'_y = \frac{dy'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \frac{dy'}{dt}\Big/\frac{dt'}{dt}$$
$$u'_z = \frac{dz'}{dt'} = \frac{dz'}{dt}\frac{dt}{dt'} = \frac{dz'}{dt}\Big/\frac{dt'}{dt}$$

Since

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \Rightarrow \frac{dt'}{dt} = \gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right) = \gamma\left(1 - \frac{vu_x}{c^2}\right)$$

Thus, we obtain

$$u'_x = \frac{\gamma\left(\frac{dx}{dt} - v\right)}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u'_y = \frac{\frac{dy}{dt}}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$$u'_z = \frac{\frac{dz}{dt}}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

EXAMPLES:

1. The relative speed of 2 with respect to 1

   $S = \text{earth}, \quad S' = 1, \quad S'' = 2$

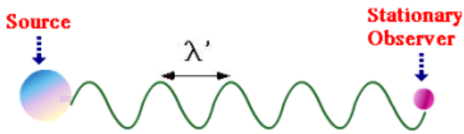
$$u_x = -0.8c$$
$$v = 0.6c$$
$$u'_x = \frac{-0.8c - 0.6c}{1 - \frac{1}{c^2}(-0.8c)(0.6c)} = -0.95c$$

2. The relative speed of 1 with respect to 2

$$\mathrm{S} = \text{earth}, \qquad \mathrm{S}' = 2, \qquad \mathrm{S}'' = 1$$
$$u_x = 0.6c$$
$$v = -0.8c$$
$$u'_x = \frac{0.6c + 0.8c}{1 - \frac{1}{c^2}(0.6c)(-0.8c)} = 0.95c$$

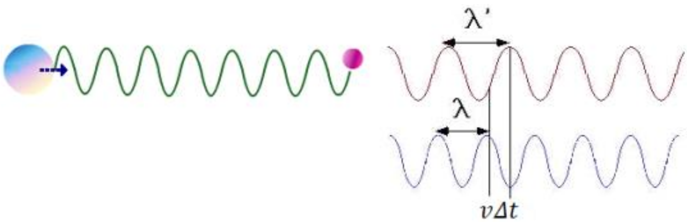
(2) Relativistic Doppler effect

1. Longitudinal



$$f_0 = \frac{c}{\lambda'} = \frac{N}{\Delta t'}$$

Considering the source approaches observer



$$\frac{c\Delta t - v\Delta t}{N} = \lambda$$
$$f = \frac{c}{\lambda} = \frac{cN}{(c-v)\Delta t} = \frac{1}{1 - \frac{v}{c}} \frac{N}{\Delta t} = \frac{1}{1 - \frac{v}{c}} \frac{f_0 \Delta t'}{\Delta t}$$

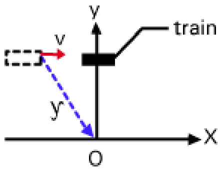
Consider time dilation: $\Delta t = \gamma \Delta t'$

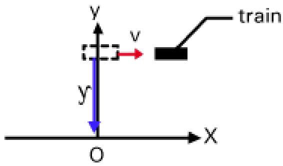
$$f = \frac{1}{1 - \frac{v}{c}} \frac{f_0}{\gamma} = f_0 \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

If $v \ll c$

$$f = f_0\left(1+\frac{v}{c}\right)^{1/2}\left(1-\frac{v}{c}\right)^{-1/2}$$
$$= f_0\left(1+\frac{v}{2c}-\cdots\right)\left(1+\frac{v}{2c}+\cdots\right)$$
$$\approx f_0\left(1+\frac{v}{c}\right)$$

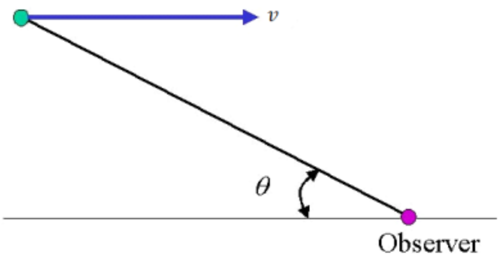
2. Transverse





$$f = \frac{N}{\Delta t} = \frac{f_0 \Delta t'}{\Delta t} = \frac{f_0}{\gamma} = f_0\sqrt{1-\frac{v^2}{c^2}}$$

3. EM Radiation Source



$$f = \frac{f_0}{\gamma}\frac{1}{1-\frac{v}{c}\cos\theta}$$

EXAMPLES:

1. A continuously emitted electromagnetic wave reflected back from a mirror with speed v.
   What is the reflected frequency?
   ANSWER:

$$f' = \sqrt{\frac{c+v}{c-v}} f_0$$

$$f = \sqrt{\frac{c+v}{c-v}} f'$$

$$f = \frac{c+v}{c-v} f_0$$

2. If $v \ll c$, we assume $f + f_0 \approx 2f_0$, what is the beat frequency?
   ANSWER:

$$f(c-v) = f_0(c+v)$$

$$(f - f_0)c = (f + f_0)v \approx 2f_0 v$$

$$f - f_0 = \frac{2f_0 v}{c} = \frac{2v}{\lambda}$$

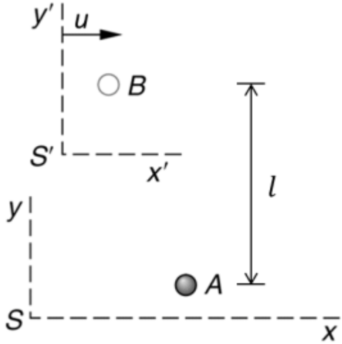
3. If beat frequency measurement is accurate to $\pm 5\ \mathrm{H_z}$, how accurate is the v measured?
   ANSWER:

$$\Delta v = \frac{\Delta(f - f_0)\lambda}{2} = 0.075\,\mathrm{m/s}$$

5-3 Relativistic Momentum and Energy

A. RELATIVISTIC MOMENTUM

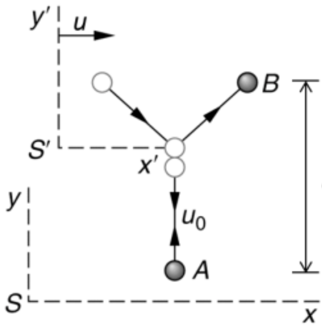
(1) Consider an elastic collision between two balls A and B. Before the collision, ball A is at rest in S frame and ball B is at rest in S′ frame. The balls are initially l apart.



Then, A is thrown up with velocity u_A while B is thrown down with velocity u_B', where

$$u_A = u_B'$$

When two balls collide, A rebounds with velocity u_A while B rebounds with velocity u_B'.



In S frame, the round-trip time Δt_0 for A is measured as,

$$\Delta t_0 = \frac{l}{u_A}$$

and the velocity u_B is found from

$$u_B = \frac{l}{\Delta t_B}$$

where Δt_B is the time required for B to make its round-trip as

measured in S frame.
Since in S′ frame, B's round-trip requires the time Δt_0, we have

$$\Delta t_{\mathrm{B}} = \frac{\Delta t_0}{\sqrt{1-u^2/c^2}}$$

Thus, we obtain

$$u_B = \frac{l}{\Delta t_B} = \frac{l}{\dfrac{\Delta t_0}{\sqrt{1-u^2/c^2}}} = \sqrt{1-u^2/c^2}\,\frac{l}{\Delta t_0}$$

$$u_A = \frac{l}{\Delta t_0}$$

(2) In S frame, use the classical definition of momentum $\vec{p} = m\vec{u}$, then we have

$$p_A = m_A u_A = m_A \frac{l}{\Delta t_0}$$

$$p_B = m_B u_B = m_B \sqrt{1-u^2/c^2}\,\frac{l}{\Delta t_0}$$

where m_A and m_B are masses as measured in S frame.
If $m_A = m_B = m$, momentum will not be conserved. Thus, if we let

$$m_B = \frac{m_A}{\sqrt{1-u^2/c^2}}$$

then momentum will be conserved.

(3) During the collision both A and B are moving in S and S′ frames. Since the mass of the ball has clear definition and been measured at rest in an inertial frame, we consider the limit cases $u_A = 0$ and $u'_B = 0$ and obtain the rest mass of A in S frame and the rest mass of B in S' frame, i.e.,

$$m_A = m$$

Now, we consider the mass of B in S frame moving at the velocity u as

$$m_B = m(u) = \frac{m}{\sqrt{1-u^2/c^2}}$$

We can define the relativistic momentum as:

$$\vec{p} = m(\vec{u})\vec{u} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

EXAMPLES:

1. Find the acceleration of a particle of mass m and velocity $\vec{v}$ when it is acted upon by the constant force $\vec{F}$, where $\vec{F}$ is parallel to $\vec{v}$.

ANSWER:

$$\begin{aligned}\vec{F} &= \frac{d\vec{p}}{dt} \\ &= \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right) \\ &= m\left[\frac{1}{\sqrt{1-u^2/c^2}} + \frac{u^2/c^2}{(1-u^2/c^2)^{3/2}}\right]\frac{d\vec{u}}{dt} \\ &= \frac{m\frac{d\vec{u}}{dt}}{(1-u^2/c^2)^{3/2}}\end{aligned}$$

B. RELATIVISTIC ENERGY

(1) The Newton's second law

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

Using work-energy theorem: The kinetic energy E_k is the work done by a net force in accelerating a particle from rest to velocity u,

$$E_k = \int \vec{F}\cdot d\vec{r} = \int \frac{d}{dt}\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)dx = \int u\, d\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)$$

Since

$$\begin{aligned}d\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right) &= \frac{m\,du}{\sqrt{1-u^2/c^2}} + \frac{1}{2}\frac{mu}{(1-u^2/c^2)^{3/2}}\frac{2u\,du}{c^2} \\ &= m\left(1-\frac{u^2}{c^2}\right)^{-3/2} du\end{aligned}$$

we obtain

$$E_k = \int_0^u um\left(1-\frac{u^2}{c^2}\right)^{-3/2} du = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right)$$

Let $E = \dfrac{mc^2}{\sqrt{1-u^2/c^2}}$

$\Rightarrow E_k = E - mc^2$

(2) The relativistic energy

$$E = E_k + mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

The square of the relativistic energy

$$E^2 = \left(E_k + mc^2\right)^2 = \frac{m^2c^4}{1 - u^2/c^2}$$

$$\Rightarrow E^2 = E_k^2 + 2E_k mc^2 + m^2c^4 = \frac{m^2c^4}{1 - u^2/c^2}$$

Here

$$E_k^2 + 2E_k mc^2 = \frac{m^2c^4}{1 - u^2/c^2} - m^2c^4 = \frac{m^2u^2c^2}{1 - u^2/c^2} = p^2c^2$$

Thus, we obtain

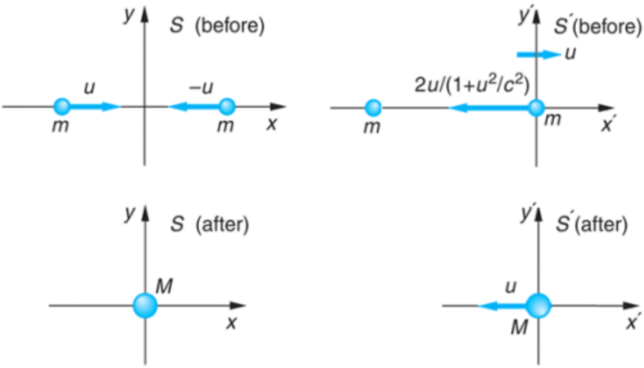
$$E^2 = E_k^2 + 2E_k mc^2 + m^2c^4 = p^2c^2 + m^2c^4$$

$$E = \sqrt{p^2c^2 + m^2c^4}$$

For a particle at rest in S frame, i.e., $u = 0$, we have

$$E = mc^2$$

(3) Conservation of energy



$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} = \frac{2u}{1 + \frac{u^2}{c^2}}$$

In S frame, before collision:

$$E_b = \frac{mc^2}{\sqrt{1 - u^2/c^2}} + \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \frac{2mc^2}{\sqrt{1 - u^2/c^2}}$$

after collision:

$$E_a = Mc^2$$

Since

$$M = 2\frac{m}{\sqrt{1 - u^2/c^2}}$$

$E_a = E_b \Rightarrow$ Conservation of energy in S frame.

In S′ frame, before collision:

$$E_b' = \frac{mc^2}{\sqrt{1 - \frac{1}{c^2}\left(\frac{2u}{1 + u^2 / c^2}\right)^2}} + mc^2 = \frac{2mc^2}{1 - u^2/c^2} = \gamma E_b$$

after collision:

$$E_a' = \frac{Mc^2}{\sqrt{1 - u^2/c^2}}$$

$E_a' = E_b' \Rightarrow$ The energy is also conserved in S' frame.

$\Rightarrow$ The energy is conserved in both S and S' frames.

EXAMPLES:

1. A stationary body explodes into two fragments each of mass 1.0 kg that move apart at speeds of 0.6c relative to the original body. Find the mass of the original body.

   ANSWER:

   The rest energy of the original body must equal the sum of the total energies of the fragments.

$$\frac{m_1c^2}{\sqrt{1 - u_1^2/c^2}} + \frac{m_2c^2}{\sqrt{1 - u_2^2/c^2}} = mc^2$$

$$\Rightarrow \frac{1.0\text{c}^2}{\sqrt{1 - (0.6)^2}} + \frac{1.0\text{c}^2}{\sqrt{1 - (0.6)^2}} = \frac{2c^2}{\sqrt{1 - (0.6)^2}} = mc^2$$

$$\Rightarrow m = \frac{2}{\sqrt{1 - (0.6)^2}} = 2.5\ \text{kg}$$

5-4 Lorentz Invariance

A. INVARIANT SPACETIME INTERVAL

(1) The spacetime interval

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

OS:

In Euclidean space (3D):

$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ is invariant under Galilean transformation.

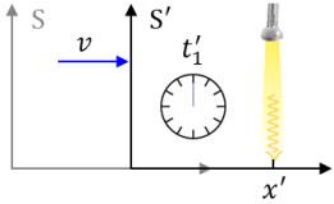
Since

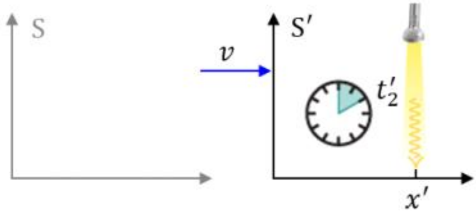
$$\begin{aligned}
(\Delta s')^2 &= (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \\
&= c^2\gamma^2\left(\Delta t - \frac{v}{c^2}\Delta x\right)^2 - \gamma^2(\Delta x - v\Delta t)^2 - (\Delta y)^2 - (\Delta z)^2 \\
&= \gamma^2\left(c^2(\Delta t)^2 - 2v\Delta t\Delta x + \frac{v^2}{c^2}(\Delta x)^2\right) \\
&\qquad - \gamma^2\left((\Delta x)^2 - 2v\Delta x\Delta t + v^2(\Delta t)^2\right) - (\Delta y)^2 - (\Delta z)^2 \\
&= \gamma^2(\Delta t)^2(c^2 - v^2) - \gamma^2\Delta x^2\left(1 - \frac{v^2}{c^2}\right) - (\Delta y)^2 - (\Delta z)^2 \\
&= c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\
&= (\Delta s)^2
\end{aligned}$$

$\Rightarrow (\Delta s)^2$ is invariant under the Lorentz transformation.

EXAMPLES:

1. Time dilation



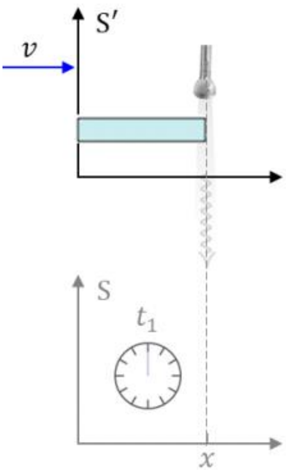


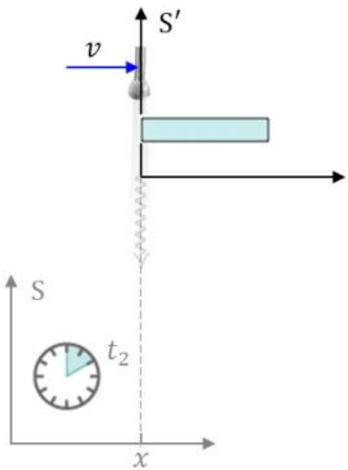
$$(\Delta s')^2 = (c\Delta t')^2$$
$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$$

Since $(\Delta s')^2 = (\Delta s)^2$

$$(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2$$
$$(c\Delta t)^2 - (v\Delta t)^2 = (c\Delta t')^2$$
$$(\Delta t)^2 = \frac{c^2}{c^2 - v^2}(\Delta t')^2$$
$$\Delta t = \sqrt{\frac{c^2}{c^2 - v^2}}\Delta t' = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta t' = \gamma \Delta t'$$

2. Length contraction





$$(\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2 = c^2\left(\frac{L'}{v}\right)^2 - L'^2$$

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 = c^2\left(\frac{L}{v}\right)^2$$

Since $(\Delta s')^2 = (\Delta s)^2$

$$c^2\frac{L'^2}{v^2} - L'^2 = c^2\frac{L^2}{v^2}$$

$$L'^2 = \frac{c^2/v^2}{c^2/v^2 - 1}L^2 = \frac{1}{1 - v^2/c^2}L^2$$

$$L' = \gamma L$$

$$L = \frac{L'}{\gamma}$$

(2) Four-vectors:

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

The Lorentz transformation

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix}$$

$$\mathbf{X}' = \mathbf{\Lambda} \cdot \mathbf{X}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma\left(x - \frac{v}{c}(ct)\right) \\ y \\ z \\ \gamma\left(ct - \frac{v}{c}x\right) \end{pmatrix}$$

Since $(\Delta s)^2 = c^2t^2 - x^2 - y^2 - z^2$, we introduce the Minkowski metric matrix $\boldsymbol{\eta}$ as,

$$\boldsymbol{\eta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, we obtain

$$\mathbf{X}^\dagger \cdot \boldsymbol{\eta} \cdot \mathbf{X} = c^2t^2 - x^2 - y^2 - z^2 = (\Delta s)^2$$

From the invariance of the spacetime interval, $(\Delta s')^2 = (\Delta s)^2$

$$\mathbf{X}'^{\dagger}\cdot\boldsymbol{\eta}\cdot\mathbf{X}'=\mathbf{X}^{\dagger}\cdot\boldsymbol{\Lambda}^{\dagger}\cdot\boldsymbol{\eta}\cdot\boldsymbol{\Lambda}\cdot\mathbf{X}=\mathbf{X}^{\dagger}\cdot\boldsymbol{\eta}\cdot\mathbf{X}$$

Thus, we obtain

$$\boldsymbol{\Lambda}^{\dagger}\cdot\boldsymbol{\eta}\cdot\boldsymbol{\Lambda}=\boldsymbol{\eta}$$

COMMENT:

$$\begin{pmatrix}\gamma & 0 & 0 & -\gamma\frac{v}{c}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\gamma\frac{v}{c} & 0 & 0 & \gamma\end{pmatrix}\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma & 0 & 0 & -\gamma\frac{v}{c}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\gamma\frac{v}{c} & 0 & 0 & \gamma\end{pmatrix}$$

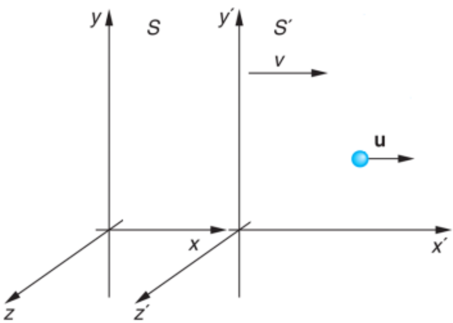
$$=\begin{pmatrix}\gamma & 0 & 0 & -\gamma\frac{v}{c}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\gamma\frac{v}{c} & 0 & 0 & \gamma\end{pmatrix}\begin{pmatrix}-\gamma & 0 & 0 & \gamma\frac{v}{c}\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ -\gamma\frac{v}{c} & 0 & 0 & \gamma\end{pmatrix}$$

$$=\begin{pmatrix}\gamma^2\left(-1+\frac{v^2}{c^2}\right) & 0 & 0 & \gamma^2\frac{v}{c}-\gamma^2\frac{v}{c}\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ \gamma^2\frac{v}{c}-\gamma^2\frac{v}{c} & 0 & 0 & \gamma^2\left(-\frac{v^2}{c^2}+1\right)\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

B. INVARIANT REST ENERGY AND REST MASS

(1) Lorentz transformation of energy and momentum



In S frame:

$$p_x = \frac{mu}{\sqrt{1-u^2/c^2}}, \qquad E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

In S' frame:

$$p'_x = \frac{mu'}{\sqrt{1-u'^2/c^2}}, \qquad E' = \frac{mc^2}{\sqrt{1-u'^2/c^2}}$$

Using

$$\frac{1}{\sqrt{1-u'^2/c^2}} = \frac{1}{\sqrt{1-\frac{1}{c^2}\left(\frac{u-v}{1-uv/c^2}\right)^2}}$$

$$= \frac{1-uv/c^2}{\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}+\frac{u^2v^2}{c^4}}}$$

$$= \frac{1-uv/c^2}{\sqrt{1-v^2/c^2}\sqrt{1-u^2/c^2}}$$

$$= \gamma\frac{1-uv/c^2}{\sqrt{1-u^2/c^2}}$$

We obtain

$$p'_x = \frac{mu'}{\sqrt{1-u'^2/c^2}}$$

$$= m\gamma\frac{1-uv/c^2}{\sqrt{1-u^2/c^2}}\left(\frac{u-v}{1-uv/c^2}\right)$$

$$= \gamma\left(\frac{mu}{\sqrt{1-u^2/c^2}} - \frac{mv}{\sqrt{1-u^2/c^2}}\right)$$

$$= \gamma\left(p_x - \frac{v}{c^2}E\right)$$

$$p'_y = p_y$$

$$p'_z = p_z$$

$$E' = \frac{mc^2}{\sqrt{1-u'^2/c^2}} = \gamma\left(\frac{mc^2}{\sqrt{1-u^2/c^2}} - \frac{mc^2(uv/c^2)}{\sqrt{1-u^2/c^2}}\right) = \gamma(E - vp_x)$$

(2) The rest energy

$$(mc^2)^2 = E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2$$

Since

$$
\begin{aligned}
(m'c^2)^2 &= E'^2 - (p'_x c)^2 - (p'_y c)^2 - (p'_z c)^2 \\
&= \gamma^2 (E - v p_x)^2 - \left(\gamma \left(p_x - \frac{v}{c^2} E\right) c\right)^2 - (p_y c)^2 - (p_z c)^2 \\
&= \gamma^2 \left(E^2 - 2Evp_x + v^2 p_x^2 - p_x^2 c^2 + 2p_x v E - \frac{v^2}{c^2} E^2\right) \\
&\qquad - (p_y c)^2 - (p_z c)^2 \\
&= \gamma^2 \left[\left(1 - \frac{v^2}{c^2}\right) E^2 - (c^2 - v^2) p_x^2\right] - (p_y c)^2 - (p_z c)^2 \\
&= E^2 - \frac{c^2 - v^2}{1 - v^2/c^2} p_x^2 - (p_y c)^2 - (p_z c)^2 \\
&= E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2 \\
&= (mc^2)^2
\end{aligned}
$$

$\Rightarrow (mc^2)^2$ is invariant under a Lorentz transformation.

(3) 4-vectors:

$$
\mathbf{P} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}
$$

The Lorentz transformation

$$
\mathbf{\Lambda} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \frac{v}{c} & 0 & 0 & \gamma \end{pmatrix}
$$

$$
\mathbf{P}' = \mathbf{\Lambda} \cdot \mathbf{P}
$$

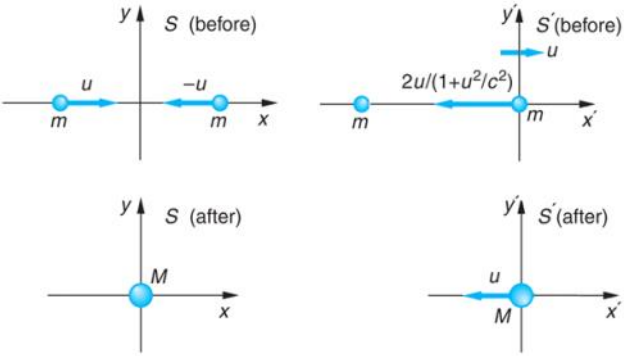
$$
\begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ E'/c \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix} = \begin{pmatrix} \gamma \left(p_x - \frac{v}{c}\frac{E}{c}\right) \\ p_y \\ p_z \\ \gamma \left(\frac{E}{c} - \frac{v}{c} p_x\right) \end{pmatrix}
$$

Since $(mc^2)^2 = E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2$, we obtain

$\mathbf{P}^\dagger \cdot \boldsymbol{\eta} \cdot \mathbf{P} = E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2$ which is invariant.

(4) The rest mass

Considering an atom composites two identical moving particles



In S frame (center-of-mass):

$$p_1 = \frac{m_1 u_1}{\sqrt{1-u^2/c^2}} = \frac{mu}{\sqrt{1-u^2/c^2}}$$

$$p_2 = \frac{m_2 u_2}{\sqrt{1-u^2/c^2}} = -\frac{mu}{\sqrt{1-u^2/c^2}}$$

The total momentum is $p = p_1 + p_2 = 0$

$$E_1 = \sqrt{\left(m_1 c^2\right)^2 + \left(p_1 c\right)^2}$$

$$= \sqrt{m^2 c^4 + \frac{m^2 u^2 c^2}{1-u^2/c^2}}$$

$$= \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

$$E_2 = \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

The total energy is

$$E = E_1 + E_2 = 2\frac{mc^2}{\sqrt{1-u^2/c^2}}$$

The rest mass is

$$m = \frac{\sqrt{E^2 - p^2 c^2}}{c^2} = \frac{E}{c^2} = \frac{2m}{\sqrt{1-u^2/c^2}}$$

In S′ frame (Lab frame):

$$p_1' = 0$$

$$p_2' = \gamma\left(p_2 - \frac{u}{c^2}E_2\right)$$
$$= \gamma\left(-\frac{mu}{\sqrt{1-u^2/c^2}} - \frac{u}{c^2}\frac{mc^2}{\sqrt{1-u^2/c^2}}\right)$$
$$= -\frac{2mu}{1-u^2/c^2}$$

The total momentum is

$$p' = p_1' + p_2' = -\frac{2mu}{1-u^2/c^2}$$

Since

$$E_1' = \gamma(E_1 - up_1)$$
$$= \gamma\left(\frac{mc^2}{\sqrt{1-u^2/c^2}} - \frac{mu}{\sqrt{1-u^2/c^2}}u\right)$$
$$= \frac{mc^2}{1-u^2/c^2}\left(1 - \frac{u^2}{c^2}\right)$$
$$= mc^2$$
$$E_2' = \gamma(E_2 - up_2)$$
$$= \gamma\left(\frac{mc^2}{\sqrt{1-u^2/c^2}} + \frac{mu}{\sqrt{1-u^2/c^2}}u\right)$$
$$= \left(\frac{1+u^2/c^2}{1-u^2/c^2}\right)mc^2$$

OS:

From the relativistic energy

$$E_1' = \sqrt{(mc^2)^2 + (p_1'c)^2} = mc^2$$
$$E_2' = \sqrt{(mc^2)^2 + (p_2'c)^2}$$
$$= \sqrt{m^2c^4 + 4\gamma^4 m^2 u^2 c^2}$$
$$= mc^2\sqrt{\frac{1 - 2\frac{u^2}{c^2} + \frac{u^4}{c^4} + 4\frac{u^2}{c^2}}{(1-u^2/c^2)^2}}$$
$$= \left(\frac{1+u^2/c^2}{1-u^2/c^2}\right)mc^2$$

The total energy is

$$E = E_1' + E_2' = mc^2 + \left(\frac{1+u^2/c^2}{1-u^2/c^2}\right)mc^2 = \frac{2mc^2}{1-u^2/c^2}$$

The rest mass is

$$m = \frac{\sqrt{E^2 - p^2c^2}}{c^2}$$

$$= \frac{\sqrt{\left(\frac{2mc^2}{1-u^2/c^2}\right)^2 - \left(-\frac{2mu}{1-u^2/c^2}\right)^2 c^2}}{c^2}$$

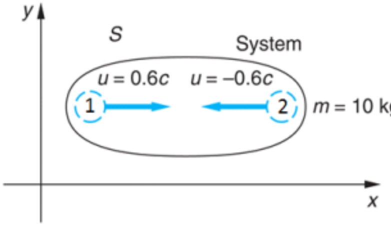
$$= \frac{2mc^2}{1-u^2/c^2}\frac{\sqrt{1-u^2/c^2}}{c^2}$$

$$= \frac{2m}{\sqrt{1-u^2/c^2}}$$

EXAMPLES:

1. Considering an atom composites two identical moving particles

   In S frame (center-of-mass):



The rest mass m of each particle is 4 kg.

$$p_{1x} = \frac{4 \times 0.6c}{\sqrt{1 - \frac{(0.6c)^2}{c^2}}} = 3c$$

$$p_{2x} = -3c$$

Total momentum: $p = p_{1x} + p_{2x} = 0$

$$E_1 = \sqrt{(mc^2)^2 + (pc)^2} = \sqrt{4^2 + 3^2}c^2 = 5c^2$$

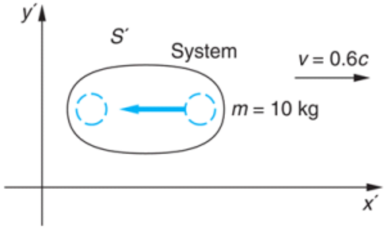
$$E_2 = 5c^2$$

Total energy: $E = E_1 + E_2 = 10c^2$

The rest mass of the atom is

$$m = \frac{\sqrt{E^2 - p^2c^2}}{c^2} = \sqrt{10^2 - 0^2} = 10$$

In S' frame (Lab frame):



$$p'_{1x} = 0$$

$$p'_{2x} = \gamma\left(p_{2x} - v\frac{E_2}{c^2}\right) = \frac{-3c - 0.6 \times 5}{0.8} = -7.5c$$

Total momentum: $p = -7.5c$

$$E'_1 = 4c^2$$

$$E'_2 = \gamma(E_2 - vp_{2x}) = \frac{5c^2 + 0.6c \times 3c}{\sqrt{1 - 0.6^2}} = 8.5c^2$$

Total energy: $E = 4c^2 + 8.5c^2 = 12.5c^2$

The rest mass of the atom is $m = \sqrt{12.5^2 - 7.5^2} = 10$

2. An unstable particle having a mass of 3.34×10^{-27} kg is initially at rest. The particle decays into two fragments that fly off with velocity of $0.987c$ and $-0.868c$. Find the rest masses of the fragments?

ANSWER:

Conservation of energy in CM frame

$$Mc^2 = \frac{m_1 c^2}{\sqrt{1 - \frac{u_1^2}{c^2}}} + \frac{m_2 c^2}{\sqrt{1 - \frac{u_2^2}{c^2}}}$$

Conservation of momentum in CM frame

$$0 = \frac{m_1 u_1}{\sqrt{1 - \frac{u_1^2}{c^2}}} - \frac{m_2 u_2}{\sqrt{1 - \frac{u_2^2}{c^2}}}$$

$$m_1 = \frac{u_2}{u_1}\frac{\sqrt{1 - \frac{u_1^2}{c^2}}}{\sqrt{1 - \frac{u_2^2}{c^2}}} m_2 = \frac{0.868c}{0.987c} \times \frac{2.01}{6.22} m_2 = 0.284 m_2$$

$$\begin{aligned} 3.34 \times 10^{-27} &= 6.22 m_1 + 2.01 m_2 \\ &= 6.22 \times 0.284 m_2 + 2.01 m_2 \\ &= 3.78 m_2 \end{aligned}$$

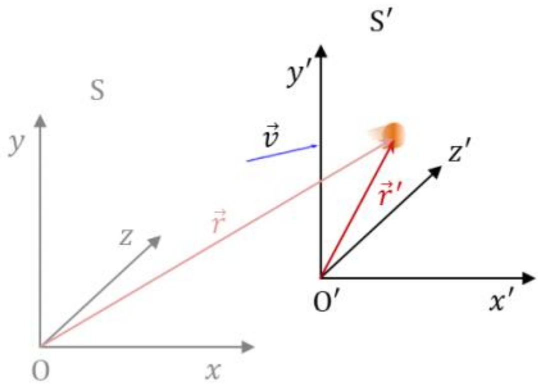
$$m_2 = 8.84 \times 10^{-28}$$

$$m_1 = 2.51 \times 10^{-28}$$

5-5 Electric Field Measured in Different Inertial Frames

A. LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS

(1) General Lorentz transformation



We decompose $\vec{r}$ into $\vec{r}_{\parallel}$ and $\vec{r}_{\perp}$ which lie parallel and perpendicular to $\vec{v}$. Thus, we obtain the Lorentz transformation of vectors:

$$\vec{r}\,'_{\perp} = \vec{r}_{\perp}$$

$$\vec{r}\,'_{\parallel} = \gamma\left(\vec{r}_{\parallel} - \frac{\vec{v}}{c}(ct)\right)$$

$$ct' = \gamma\left(ct - \frac{\vec{v}}{c}\cdot\vec{r}\right)$$

(2) Consider the electromagnetic fields [c.f.7-2]

In S frame:

$$\vec{E} = -\nabla\varphi - \frac{\partial}{\partial t}\vec{A}, \qquad \vec{B} = \nabla\times\vec{A}$$

In S′ frame:

$$\vec{E}' = -\nabla'\varphi' - \frac{\partial}{\partial t'}\vec{A}', \qquad \vec{B}' = \nabla'\times\vec{A}'$$

We can decompose $\vec{E}'$ and $\vec{B}'$ into $\vec{E}'_{\parallel}$, $\vec{E}'_{\perp}$, $\vec{B}'_{\parallel}$, and $\vec{B}'_{\perp}$ which lie parallel and perpendicular to $\vec{v}$. Thus, we obtaint the parallel and perpendicular components of $\vec{E}'$ are

$$\vec{E}'_{\parallel} = -\nabla'_{\parallel}\varphi' - \frac{\partial}{\partial t'}\vec{A}'_{\parallel}$$

$$\vec{E}'_\perp = -\nabla'_\perp \varphi' - \frac{\partial}{\partial t'}\vec{A}'_\perp$$

and the parallel and perpendicular components of $\vec{B}'$ are

$$\begin{aligned}\vec{B}' &= \nabla' \times \vec{A}' \\ &= \left(\nabla'_\parallel + \nabla'_\perp\right) \times \left(\vec{A}'_\parallel + \vec{A}'_\perp\right) \\ &= \underbrace{\nabla'_\parallel \times \vec{A}'_\parallel + \nabla'_\perp \times \vec{A}'_\perp}_{\vec{B}'_\parallel} + \underbrace{\nabla'_\parallel \times \vec{A}'_\perp + \nabla'_\perp \times \vec{A}'_\parallel}_{\vec{B}'_\perp}\end{aligned}$$

$$\vec{B}'_\parallel = \left(\nabla' \times \vec{A}'\right)_\parallel = \underbrace{\nabla'_\parallel \times \vec{A}'_\parallel}_{=0} + \nabla'_\perp \times \vec{A}'_\perp = \nabla'_\perp \times \vec{A}'_\perp$$

$$\vec{B}'_\perp = \left(\nabla' \times \vec{A}'\right)_\perp = \nabla'_\parallel \times \vec{A}'_\perp + \nabla'_\perp \times \vec{A}'_\parallel$$

Here, we would to introduce the Lorentz transformation of ∇, φ, and $\vec{A}$.

The Lorentz transformation of ∇:

$$\frac{\partial}{\partial \vec{r}'_\perp} = \frac{\partial}{\partial \vec{r}_\perp}$$

$$\frac{\partial}{\partial \vec{r}'_\parallel} = \frac{\partial \vec{r}_\parallel}{\partial \vec{r}'_\parallel}\frac{\partial}{\partial \vec{r}_\parallel} + \frac{\partial \vec{r}_\perp}{\partial \vec{r}'_\parallel}\frac{\partial}{\partial \vec{r}_\perp} + \frac{\partial t}{\partial \vec{r}'_\parallel}\frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial \vec{r}_\parallel} + \frac{\vec{v}}{c}\frac{\partial}{c\partial t}\right)$$

$$\frac{\partial}{c\partial t'} = \frac{\partial t}{c\partial t'}\frac{\partial}{\partial t} + \frac{\partial \vec{r}}{c\partial t'} \cdot \frac{\partial}{\partial \vec{r}} = \gamma\left(\frac{\partial}{c\partial t} + \frac{\vec{v}}{c} \cdot \frac{\partial}{\partial \vec{r}}\right)$$

Expressed by the symbol ∇ (nabla)

$$\nabla'_\perp = \nabla_\perp$$

$$\nabla'_\parallel = \gamma\left(\nabla_\parallel + \frac{\vec{v}}{c}\frac{\partial}{c\partial t}\right)$$

$$\frac{\partial}{c\partial t'} = \gamma\left(\frac{\partial}{c\partial t} + \frac{\vec{v}}{c} \cdot \nabla\right) \Rightarrow \frac{\partial}{\partial t'} = \gamma\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right)$$

The Lorentz transformation of φ and $\vec{A}$:

$$\begin{matrix} A_x & \leftrightarrow x \\ A_y & \leftrightarrow y \\ A_z & \leftrightarrow z \\ \varphi/c & \leftrightarrow ct \end{matrix} \Rightarrow \begin{cases} \vec{A}'_\perp = \vec{A}_\perp \\ \vec{A}'_\parallel = \gamma\left(\vec{A}_\parallel - \frac{\vec{v}}{c}(\varphi/c)\right) = \gamma\left(\vec{A}_\parallel - \frac{\vec{v}}{c^2}\varphi\right) \\ \varphi'/c = \gamma\left((\varphi/c) - \frac{\vec{v}}{c} \cdot \vec{A}\right) \Rightarrow \varphi' = \gamma\left(\varphi - \vec{v} \cdot \vec{A}\right) \end{cases}$$

(3) The Lorentz transformation of $\vec{B}'_\parallel$

$$\vec{B}'_\parallel = \nabla'_\perp \times \vec{A}'_\perp$$

Since

$$\nabla'_\perp = \nabla_\perp$$

$$\vec{A}'_\perp = \vec{A}_\perp$$

thus we obtain

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

(4) The Lorentz transformation of $\vec{B}'_{\perp}$

$$\vec{B}'_{\perp} = \nabla'_{\parallel} \times \vec{A}'_{\perp} + \nabla'_{\perp} \times \vec{A}'_{\parallel}$$

Since

$$\nabla'_{\parallel} = \gamma\left(\nabla_{\parallel} + \frac{\vec{v}}{c}\frac{\partial}{c\partial t}\right)$$

$$\vec{A}'_{\parallel} = \gamma\left(\vec{A}_{\parallel} - \frac{\vec{v}}{c^2}\varphi\right)$$

thus we obtain

$$\begin{aligned}
\vec{B}'_{\perp} &= \gamma\left(\nabla_{\parallel} + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\right) \times \vec{A}_{\perp} + \nabla_{\perp} \times \gamma\left(\vec{A}_{\parallel} - \frac{\vec{v}}{c^2}\varphi\right) \\
&= \gamma\left(\nabla_{\parallel} \times \vec{A}_{\perp} + \nabla_{\perp} \times \vec{A}_{\parallel}\right) + \frac{\gamma}{c^2}\left[\vec{v} \times \frac{\partial \vec{A}_{\perp}}{\partial t} - \nabla_{\perp} \times (\vec{v}\varphi)\right] \\
&= \gamma\vec{B}_{\perp} + \frac{\gamma}{c^2}\left[-\vec{v} \times \left(-\frac{\partial \vec{A}_{\perp}}{\partial t}\right) + \vec{v} \times \nabla_{\perp}\varphi\right] \\
&= \gamma\vec{B}_{\perp} - \frac{\gamma}{c^2}\vec{v} \times \left(-\frac{\partial \vec{A}_{\perp}}{\partial t} - \nabla_{\perp}\varphi\right) \\
&= \gamma\vec{B}_{\perp} - \frac{\gamma}{c^2}\vec{v} \times \vec{E}_{\perp} \\
&= \gamma\left(\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E}\right)_{\perp}
\end{aligned}$$

(5) The Lorentz transformation of $\vec{E}'_{\parallel}$

$$\begin{aligned}
\vec{E}'_{\parallel} &= -\nabla'_{\parallel}\varphi' - \frac{\partial}{\partial t'}\vec{A}'_{\parallel} \\
&= -\gamma\left(\nabla_{\parallel} + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\right)\gamma\left(\varphi - \vec{v} \cdot \vec{A}\right) - \gamma\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right)\gamma\left(\vec{A}_{\parallel} - \frac{\vec{v}}{c^2}\varphi\right) \\
&= -\gamma^2\left(\nabla_{\parallel}\varphi + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\varphi - \nabla_{\parallel}\left(\vec{v} \cdot \vec{A}\right) - \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\left(\vec{v} \cdot \vec{A}\right)\right) \\
&\quad - \gamma^2\left(\frac{\partial}{\partial t}\vec{A}_{\parallel} + (\vec{v} \cdot \nabla)\vec{A}_{\parallel} - \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\varphi - (\vec{v} \cdot \nabla)\frac{\vec{v}}{c^2}\varphi\right)
\end{aligned}$$

Since $\vec{v}$ is a constant vector

$$\nabla_\parallel \underbrace{\left(\vec{v}\cdot\vec{A}\right)}_{\substack{\text{inner}\\\text{product}}} = \nabla_\parallel \left(\vec{v}\cdot\vec{A}_\parallel\right)$$

$$\underbrace{\left(\vec{v}\cdot\nabla\right)}_{\substack{\text{inner}\\\text{product}}}\vec{A}_\parallel = \left(\vec{v}\cdot\nabla_\parallel\right)\vec{A}_\parallel$$

thus, we obtain

$$\begin{aligned}
\vec{E}'_\parallel &= -\gamma^2\left(\nabla_\parallel\varphi + \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\varphi - \nabla_\parallel\left(\vec{v}\cdot\vec{A}_\parallel\right) - \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\left(\vec{v}\cdot\vec{A}_\parallel\right)\right)\\
&\qquad -\gamma^2\left(\frac{\partial}{\partial t}\vec{A}_\parallel + \left(\vec{v}\cdot\nabla_\parallel\right)\vec{A}_\parallel - \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\varphi - \left(\vec{v}\cdot\nabla_\parallel\right)\frac{\vec{v}}{c^2}\varphi\right)\\
&= -\gamma^2\left(\nabla_\parallel\varphi - \frac{v^2}{c^2}\nabla_\parallel\varphi\right) - \gamma^2\left(\frac{\partial}{\partial t}\vec{A}_\parallel - \frac{v^2}{c^2}\frac{\partial}{\partial t}\vec{A}_\parallel\right)\\
&\qquad +\gamma^2\left(\nabla_\parallel\left(\vec{v}\cdot\vec{A}_\parallel\right) - \left(\vec{v}\cdot\nabla_\parallel\right)\vec{A}_\parallel\right)\\
&= -\gamma^2\left(1-\frac{v^2}{c^2}\right)\nabla_\parallel\varphi - \gamma^2\left(1-\frac{v^2}{c^2}\right)\frac{\partial}{\partial t}\vec{A}_\parallel\\
&\qquad +\gamma^2\left(\nabla_\parallel\left(\vec{v}\cdot\vec{A}_\parallel\right) - \left(\vec{v}\cdot\nabla_\parallel\right)\vec{A}_\parallel\right)\\
&= -\nabla_\parallel\varphi - \frac{\partial}{\partial t}\vec{A}_\parallel + \gamma^2\left(\nabla_\parallel\left(\vec{v}\cdot\vec{A}_\parallel\right) - \left(\vec{v}\cdot\nabla_\parallel\right)\vec{A}_\parallel\right)
\end{aligned}$$

Using vector triple product identity BAC-CAB:

$$\vec{v}\times\underbrace{\left(\nabla_\parallel\times\vec{A}_\parallel\right)}_{=0} = \nabla_\parallel\left(\vec{v}\cdot\vec{A}_\parallel\right) - \left(\vec{v}\cdot\nabla_\parallel\right)\vec{A}_\parallel = 0$$

Thus, we obtain

$$\vec{E}'_\parallel = -\nabla_\parallel\varphi - \frac{\partial}{\partial t}\vec{A}_\parallel = \vec{E}_\parallel$$

(6) The Lorentz transformation of $\vec{E}'_\perp$

$$\begin{aligned}
\vec{E}'_\perp &= -\nabla'_\perp\varphi' - \frac{\partial}{\partial t'}\vec{A}'_\perp\\
&= -\nabla_\perp\gamma\left(\varphi - \vec{v}\cdot\vec{A}_\parallel\right) - \gamma\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_\parallel\right)\vec{A}_\perp\\
&= \gamma\left(-\nabla_\perp\varphi - \frac{\partial}{\partial t}\vec{A}_\perp\right) + \gamma\left(\nabla_\perp\left(\vec{v}\cdot\vec{A}_\parallel\right) - \left(\vec{v}\cdot\nabla_\parallel\right)\vec{A}_\perp\right)
\end{aligned}$$

Using vector triple product identity BAC-CAB:

$$\vec{v}\times\left(\nabla_\perp\times\vec{A}_\parallel\right) = \nabla_\perp\left(\vec{v}\cdot\vec{A}_\parallel\right) - \underbrace{\left(\vec{v}\cdot\nabla_\perp\right)}_{=0}\vec{A}_\parallel = \nabla_\perp\left(\vec{v}\cdot\vec{A}_\parallel\right)$$

$$\vec{v}\times\left(\nabla_{\parallel}\times\vec{A}_{\perp}\right)=\nabla_{\parallel}\underbrace{\left(\vec{v}\cdot\vec{A}_{\perp}\right)}_{=0}-(\vec{v}\cdot\nabla_{\parallel})\vec{A}_{\perp}=-(\vec{v}\cdot\nabla_{\parallel})\vec{A}_{\perp}$$

Thus, we obtain

$$\begin{aligned}
\vec{E}'_{\perp}&=\gamma\left(-\nabla_{\perp}\varphi-\frac{\partial}{\partial t}\vec{A}_{\perp}\right)+\gamma\left(\vec{v}\times\left(\nabla_{\perp}\times\vec{A}_{\parallel}\right)+\vec{v}\times\left(\nabla_{\parallel}\times\vec{A}_{\perp}\right)\right)\\
&=\gamma\left(-\nabla_{\perp}\varphi-\frac{\partial}{\partial t}\vec{A}_{\perp}\right)+\gamma\left(\vec{v}\times\left(\nabla\times\vec{A}\right)_{\perp}\right)\\
&=\gamma\vec{E}_{\perp}+\gamma\left(\vec{v}\times\vec{B}_{\perp}\right)\\
&=\gamma\left(\vec{E}+\vec{v}\times\vec{B}\right)_{\perp}
\end{aligned}$$

(7) Thus, we found the Lorentz transformation of electromagnetic fields

$$\vec{E}'_{\parallel}=\vec{E}_{\parallel},\qquad \vec{E}'_{\perp}=\gamma\left(\vec{E}+\vec{v}\times\vec{B}\right)_{\perp}$$

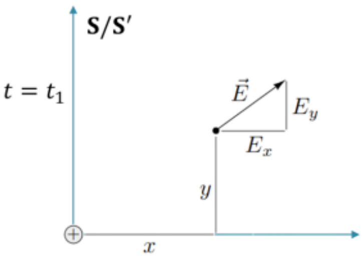
$$\vec{B}'_{\parallel}=\vec{B}_{\parallel},\qquad \vec{B}'_{\perp}=\gamma\left(\vec{B}-\frac{\vec{v}}{c^2}\times\vec{E}\right)_{\perp}$$

Suppose S frame exists in which $\vec{B}=0$ in some region. Then in any other S′ frame that moves with velocity $\vec{v}$ relative to S frame, we have

$$\boxed{\begin{aligned}\vec{E}'_{\parallel}&=\vec{E}_{\parallel},\qquad &\vec{E}'_{\perp}&=\gamma\vec{E}_{\perp}\\ \vec{B}'_{\parallel}&=0\ ,\qquad &\vec{B}'_{\perp}&=-\gamma\frac{\vec{v}}{c^2}\times\vec{E}_{\perp}\end{aligned}}\ \cdots\cdots\text{(a)}$$

B. GAUSS'S LAW FOR MOVING CHARGES

(1) A moving point charge q



We consider the fields seen by an observer in the S frame when a point charge q moves with a velocity v. The charge is at rest in S' frame, i.e., the field of the point charge in its own frame is entirely electrostatic:

$$E'_x = \frac{q}{4\pi\epsilon_0 r'^2}\cos\theta = \frac{q}{4\pi\epsilon_0}\frac{x'}{\left(x'^2+y'^2\right)^{3/2}}$$

$$E'_y = \frac{q}{4\pi\epsilon_0 r'^2}\sin\theta = \frac{q}{4\pi\epsilon_0}\frac{y'}{\left(x'^2+y'^2\right)^{3/2}}$$

Since $\vec{B}' = 0$ in S' frame, we have, according to equation (a),

$$\vec{E}_\parallel = \vec{E}'_\parallel, \qquad \vec{E}_\perp = \gamma\vec{E}'_\perp$$

Thus, we obtain

$$E_x = E'_x = \frac{q}{4\pi\epsilon_0}\frac{x'}{\left(x'^2+y'^2\right)^{3/2}}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\epsilon_0}\frac{\gamma y'}{\left(x'^2+y'^2\right)^{3/2}}$$

Using Lorentz transformation and let $t = 0$:

$$x' = \gamma(x - vt) = \gamma x$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) = -\gamma\frac{v}{c^2}x$$

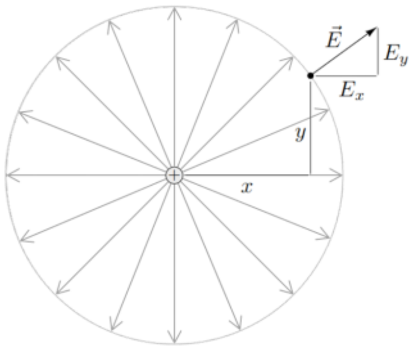
The fields become

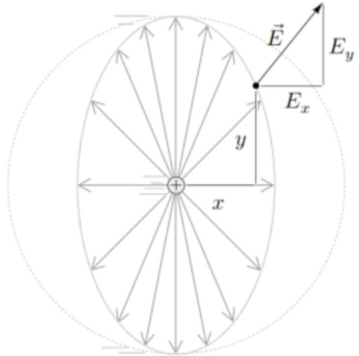
$$E_x = \frac{q}{4\pi\epsilon_0}\frac{x'}{\left(x'^2+y'^2\right)^{3/2}} = \frac{q}{4\pi\epsilon_0}\frac{\gamma x}{\left(\gamma^2x^2+y^2\right)^{3/2}}$$

$$E_y = \frac{q}{4\pi\epsilon_0}\frac{\gamma y'}{\left(x'^2+y'^2\right)^{3/2}} = \frac{q}{4\pi\epsilon_0}\frac{\gamma y}{\left(\gamma^2x^2+y^2\right)^{3/2}}$$

We obtain the electric field in the laboratory frame:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma x}{(\gamma^2 x^2 + y^2)^{3/2}}\hat{x} + \frac{q}{4\pi\epsilon_0}\frac{\gamma y}{(\gamma^2 x^2 + y^2)^{3/2}}\hat{y}$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2(x^2 + y^2/\gamma^2)^{3/2}}(x\hat{x} + y\hat{y})$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{(R^2\cos^2\theta + (1 - v^2/c^2)R^2\sin^2\theta)^{3/2}}\vec{R}$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{(1 - v^2/c^2\sin^2\theta)^{3/2}}\frac{\hat{R}}{R^2} \cdots\cdots \text{(b)}$$





The electric field is radial, but the lines of force are isotropically distributed only for $v = 0$. Along the direction of motion $(\theta = 0)$, the field strength is down by a factor $1/\gamma^2$ relative to isotropy, while in the transverse directions $(\theta = \pi/2)$ it is larger by a factor of γ.

OS:

Further reading: accelerating charge
https://physics.stackexchange.com/questions/296904/electric-field-associated-with-moving-charge

(2) Gauss's law for a moving point charge

$$d\Phi = \vec{E}\cdot d\vec{a} = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{(1 - v^2/c^2\sin^2\theta)^{3/2}}\frac{\hat{R}}{R^2}\cdot R^2\sin\theta\, d\theta d\phi\hat{R}$$
$$\Phi = \oint_S d\Phi = \int_0^\pi \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{(1 - v^2/c^2\sin^2\theta)^{3/2}}\sin\theta\, d\theta \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

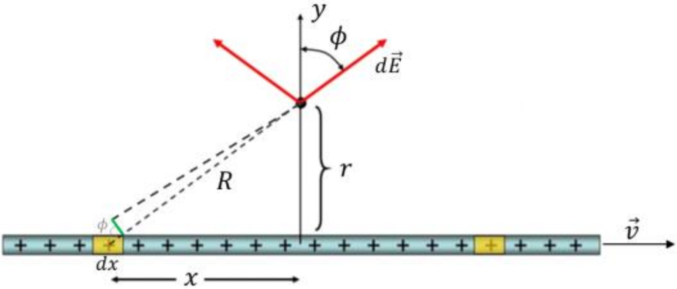
Using the integral formula:

$$\int \frac{\sin\theta\, d\theta}{(1 - a^2\sin^2\theta)^{3/2}} = \frac{-\cos\theta}{(1 - a^2)(1 - a^2\sin^2\theta)^{1/2}}$$

We obtain

$$\Phi = \frac{q(1-v^2/c^2)}{4\pi\epsilon_0} \frac{-\cos\theta}{(1-v^2/c^2)(1-v^2/c^2\sin^2\theta)^{1/2}}\Bigg|_0^\pi \cdot 2\pi$$
$$= \frac{q(1-v^2/c^2)}{2\epsilon_0} \frac{2}{(1-v^2/c^2)}$$
$$= \frac{q}{\epsilon_0}$$

(3) A line of moving charges (moving rod)



An infinity long rod carrying uniform charge λ moving with velocity v.
From equation (b), we have

$$\phi = \frac{\pi}{2} - \theta$$
$$d\vec{E} = \frac{dq}{4\pi\epsilon_0} \frac{1-v^2/c^2}{(1-v^2/c^2\sin^2\theta)^{3/2}} \frac{\hat{R}}{R^2}$$
$$= \frac{dq}{4\pi\epsilon_0} \frac{1-v^2/c^2}{(1-v^2/c^2\cos^2\phi)^{3/2}} \frac{\hat{R}}{R^2}$$
$$\vec{E} = \int_{-\pi/2}^{\pi/2} d\vec{E}\cos\phi = \int_{-\pi/2}^{\pi/2} \frac{dq}{4\pi\epsilon_0} \frac{1-v^2/c^2}{(1-v^2/c^2\sin^2\theta)^{3/2}} \frac{\hat{r}}{R^2}\cos\phi$$

Since

$$\left.\begin{array}{c} dq = \lambda dx \\ Rd\phi = dx\cos\phi \\ r = R\cos\phi \end{array}\right\} \Rightarrow dq = \lambda R\frac{d\phi}{\cos\phi} = \lambda\frac{r}{\cos\phi}\frac{d\phi}{\cos\phi} = \frac{\lambda r d\phi}{\cos^2\phi}$$

Thus, we obtain

$$\vec{E} = \int_{-\pi/2}^{\pi/2} \frac{1}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2/c^2 \cos^2\phi\right)^{3/2}} \frac{\hat{r}}{r^2/\cos^2\phi} \cos\phi \frac{\lambda r d\phi}{\cos^2\phi}$$
$$= \int_{-\pi/2}^{\pi/2} \frac{\lambda}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2/c^2 \cos^2\phi\right)^{3/2}} \frac{\hat{r}}{r} \cos\phi \, d\phi$$
$$= \int_{-\pi/2}^{\pi/2} \frac{\lambda}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2/c^2 + v^2/c^2 \sin^2\phi\right)^{3/2}} \frac{\hat{r}}{r} \frac{c}{v} d\left(\frac{v}{c}\sin\phi\right)$$

Using the integral formula:

$$\int \frac{dt}{(a^2 + t^2)^{3/2}} = \frac{t}{a^2 (a^2 + t^2)^{1/2}}$$

We obtain

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0} \frac{\hat{r}}{r} \frac{c}{v} \frac{\left(1 - v^2/c^2\right)\frac{v}{c}\sin\phi}{(1 - v^2/c^2)\left(1 - v^2/c^2 + v^2/c^2 \sin^2\phi\right)^{1/2}} \Bigg|_{-\pi/2}^{\pi/2}$$
$$= \frac{\lambda}{4\pi\epsilon_0} \frac{\hat{r}}{r} \frac{\left(1 - v^2/c^2\right)2}{(1 - v^2/c^2)(1 - v^2/c^2 + v^2/c^2)^{1/2}}$$
$$= \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

(4) Gauss's law for a line of moving charges

The field of the infinity long rod in its own frame is entirely electrostatic, i.e.,

$$\oint_S \vec{E}' \cdot d\vec{a}' = E' \cdot 2\pi r' l' = \frac{Q}{\epsilon_0} \Rightarrow \vec{E}' = \frac{Q}{2\pi\epsilon_0 r' l'} = \frac{\lambda'}{2\pi\epsilon_0 r'} \hat{r}'$$

Since $\vec{B}' = 0$ in S' frame, we have, according to equation (a),

$$r = r'$$
$$\vec{E}_\parallel = \vec{E}'_\parallel, \qquad \vec{E}_\perp = \gamma \vec{E}'_\perp$$

Thus, we obtain

$$\vec{E} = \gamma \vec{E}' = \frac{\gamma\lambda'}{2\pi\epsilon_0 r} \hat{r} \stackrel{?}{=} \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

Since S' frame moves with velocity v with respect to S frame, the distance between charges in the rod as seen in S frame is contracted by

$$l = \frac{l'}{\gamma}$$

The linear density of positive charge in this frame is correspondingly

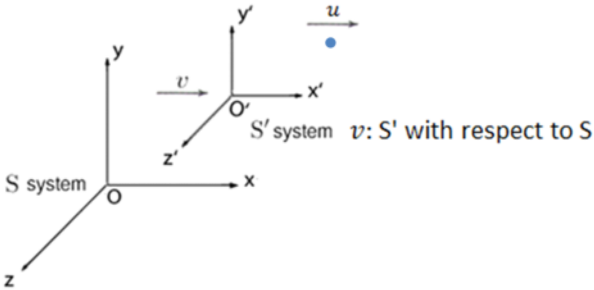
$$\lambda = \frac{Q}{l} = \frac{Q}{l'/\gamma} = \gamma \frac{Q}{l'} = \gamma\lambda' \Rightarrow \lambda' = \frac{\lambda}{\gamma}$$

Thus, we obtain

$$\vec{E} = \frac{\gamma \frac{\lambda}{\gamma}}{2\pi\epsilon_0 r}\hat{r} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

C. LORENTZ FORCE BETWEEN MOVING TEST CHARGE AND OTHER CHARGES

(1) The Lorentz transformation of the force



$$\vec{F}'_{\parallel} = \frac{d\vec{p}'_{\parallel}}{dt'} = \frac{d\vec{p}'_{\parallel}}{dt}\frac{dt}{dt'}$$
$$\vec{F}'_{\perp} = \frac{d\vec{p}'_{\perp}}{dt'} = \frac{d\vec{p}'_{\perp}}{dt}\frac{dt}{dt'}$$

Since

$$t' = \gamma\left(t - \frac{\vec{v}}{c^2}\cdot\vec{r}\right) \Rightarrow \frac{dt'}{dt} = \gamma\left(1 - \frac{\vec{v}\cdot\vec{u}}{c^2}\right)$$

Thus, we obtain

$$\frac{d\vec{p}'_{\parallel}}{dt'} = \frac{d\vec{p}'_{\parallel}}{dt}\bigg/\frac{dt'}{dt} = \frac{\gamma\left(\frac{d\vec{p}_{\parallel}}{dt} - \frac{\vec{v}}{c^2}\frac{dE}{dt}\right)}{\gamma\left(1 - \frac{\vec{v}\cdot\vec{u}}{c^2}\right)} = \frac{\frac{d\vec{p}_{\parallel}}{dt} - \frac{\vec{v}}{c^2}\left(\vec{F}\cdot\vec{u}\right)}{\left(1 - \frac{\vec{v}\cdot\vec{u}}{c^2}\right)}$$

$$\frac{d\vec{p}'_{\perp}}{dt'} = \frac{d\vec{p}'_{\perp}}{dt}\bigg/\frac{dt'}{dt} = \frac{\frac{d\vec{p}_{\perp}}{dt}}{\gamma\left(1 - \frac{\vec{v}\cdot\vec{u}}{c^2}\right)}$$

From the work-energy theorem,

$$E = \int \vec{F}\cdot d\vec{r} = \int \frac{d\vec{p}}{dt}\cdot d\vec{r} = \int \frac{d\vec{p}}{dt}\cdot\frac{d\vec{r}}{dt}dt = \int \frac{d\vec{p}}{dt}\cdot\vec{u}\,dt$$

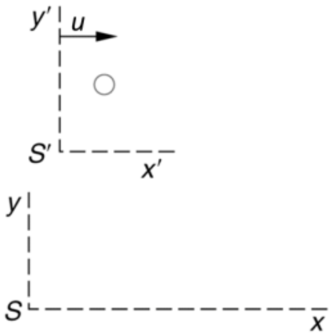
we have

$$\frac{dE}{dt} = \vec{F}\cdot\vec{u} = \frac{d\vec{p}}{dt}\cdot\vec{u} = \left(\frac{d\vec{p}_{\parallel}}{dt} + \frac{d\vec{p}_{\perp}}{dt}\right)\cdot\vec{u}$$

Thus, we obtain

$$\frac{d\vec{p}'_{\parallel}}{dt'} = \frac{\frac{d\vec{p}_{\parallel}}{dt} - \frac{\vec{v}}{c^2}\left(\frac{d\vec{p}_{\parallel}}{dt} + \frac{d\vec{p}_{\perp}}{dt}\right)\cdot\vec{u}}{\left(1 - \frac{\vec{v}\cdot\vec{u}}{c^2}\right)} = \frac{d\vec{p}_{\parallel}}{dt} - \frac{\frac{\vec{v}}{c^2}\left(\frac{d\vec{p}_{\perp}}{dt}\cdot\vec{u}\right)}{\left(1 - \frac{\vec{v}\cdot\vec{u}}{c^2}\right)}$$

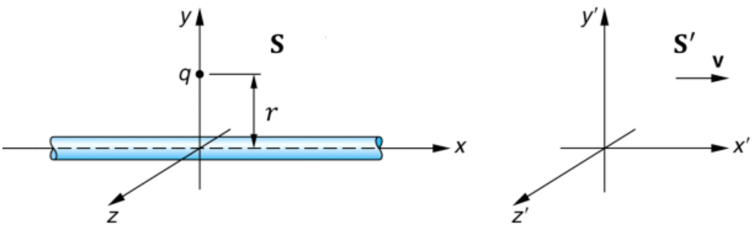
If the velocity at S frame vanishes at a given moment (namely, S frame is the rest frame), then $\vec{u} = 0$ and we have



$$\vec{F}'_{\parallel} = \frac{d\vec{p}'_{\parallel}}{dt'} = \vec{F}_{\parallel}$$

$$\vec{F}'_{\perp} = \frac{d\vec{p}'_{\perp}}{dt'} = \frac{\vec{F}_{\perp}}{\gamma}$$

(2) Stationary rod and stationary test charge q



A test charge q at rest a distance r from an infinity long rod carrying uniform charge λ. The field of the infinity long rod in its own frame is entirely electrostatic, i.e.,

$$\oint_{S} \vec{E}\cdot d\vec{a} = E\cdot 2\pi rl = \frac{Q}{\epsilon_0} \Rightarrow \vec{E} = \frac{Q}{2\pi\epsilon_0 rl} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

The Lorentz force on the test charge is

$$\vec{F} = q\vec{E} = \frac{q\lambda}{2\pi\epsilon_0 r}\hat{r}$$

Now consider an observer moving to the right with velocity v. In observer frame, both the test charge and the rod move to the right. Since $\vec{B} = 0$ in S frame, we have, according to equation (a),

$$r' = r$$
$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \qquad \vec{E}'_{\perp} = \gamma \vec{E}_{\perp}$$
$$\vec{B}'_{\parallel} = 0\ , \qquad \vec{B}'_{\perp} = -\gamma \frac{\vec{v}}{c^2} \times \vec{E}_{\perp}$$

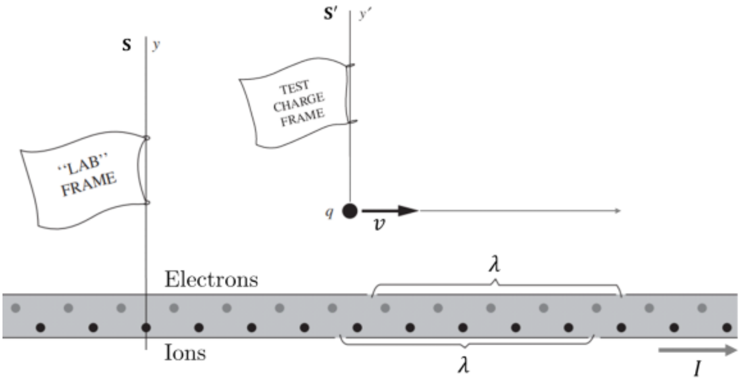
Thus, we obtain

$$\vec{E}' = \gamma \vec{E} = \frac{\gamma\lambda}{2\pi\epsilon_0 r'}\hat{r}' \quad \cdots\cdots \text{ since parallel components canceled out}$$
$$\vec{B}' = -\gamma \frac{\vec{v}}{c^2} \times \vec{E} = -\frac{\gamma\lambda}{2\pi\epsilon_0 c^2 r'}\vec{v} \times \hat{r}' = -\frac{\gamma\lambda v}{2\pi\epsilon_0 c^2 r'}\hat{z}$$

and the Lorentz force is

$$\begin{aligned}\vec{F}' &= q\vec{E}' + q\vec{v} \times \vec{B}' \\ &= \frac{\gamma q\lambda}{2\pi\epsilon_0 r'}\hat{r}' - \frac{\gamma q\lambda v^2}{2\pi\epsilon_0 c^2 r'}\hat{r}' \\ &= \frac{\gamma q\lambda}{2\pi\epsilon_0 r'}\left(1 - \frac{v^2}{c^2}\right)\hat{r}' \\ &= \frac{q\lambda}{2\pi\epsilon_0 \gamma r'}\hat{r}' \\ &= \frac{\vec{F}}{\gamma}\end{aligned}$$

(3) The Lorentz force between a moving charge and other moving charges.



In the test charge frame, the positive ions are moving with velocity $-v$. The distance between ions as seen in the test charge frame is contracted by

$$l' = \frac{l}{\gamma}$$

The linear density of **positive ions** is correspondingly

$$\lambda' = \frac{Q}{l'} = \frac{Q}{l/\gamma} = \gamma\frac{Q}{l} = \gamma\lambda$$

Electrons are already moving with velocity $I = \lambda v_0$ in the lab frame. Using addition of velocities, the velocity of electrons in the test charge frame is

$$v_0' = \frac{v_0 - v}{1 - \frac{v_0 v}{c^2}}$$

The Lorentz factor of electrons in the test charge frame is

$$\gamma_0' = \frac{1}{\sqrt{1 - \frac{v_0'^2}{c^2}}}, \qquad \gamma_0 = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}$$

Thus, the linear density of **electrons** is correspondingly

$$-\gamma_0'\frac{\lambda}{\gamma_0}$$

Since

$$\begin{aligned}
\frac{1}{\sqrt{1 - \frac{v_0'^2}{c^2}}} &= \frac{1}{\sqrt{1 - \frac{1}{c^2}\left(\frac{v_0 - v}{1 - \frac{v_0 v}{c^2}}\right)^2}} \\
&= \frac{1}{\sqrt{\left(1 - \frac{v_0 v}{c^2}\right)^2 - \left(\frac{v_0}{c} - \frac{v}{c}\right)^2}}\left(1 - \frac{v_0 v}{c^2}\right) \\
&= \frac{1}{\sqrt{1 - 2\frac{v_0 v}{c^2} + \frac{v_0^2 v^2}{c^4} - \frac{v_0^2}{c^2} + 2\frac{v_0 v}{c^2} - \frac{v^2}{c^2}}}\left(1 - \frac{v_0 v}{c^2}\right) \\
&= \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2} - \frac{v^2}{c^2} + \frac{v_0^2 v^2}{c^4}}}\left(1 - \frac{v_0 v}{c^2}\right) \\
&= \underbrace{\frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}}_{=\gamma_0}\underbrace{\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}_{=\gamma}\left(1 - \frac{v_0 v}{c^2}\right)
\end{aligned}$$

we obtain

$$-\gamma_0'\frac{\lambda}{\gamma_0} = -\gamma_0\gamma\left(1 - \frac{v_0 v}{c^2}\right)\frac{\lambda}{\gamma_0} = -\gamma\lambda\left(1 - \frac{v_0 v}{c^2}\right)$$

The total linear density of charge in the test charge frame is

$$\lambda' = \gamma\lambda - \gamma_0'\frac{\lambda}{\gamma_0} = \gamma\lambda - \gamma\lambda\left(1 - \frac{v_0 v}{c^2}\right) = \gamma\lambda\frac{v_0 v}{c^2}$$

The field of the wire in the test charge frame is

$$\vec{E}' = -\frac{\lambda'}{2\pi\epsilon_0 r'}\hat{r}' = -\frac{\gamma\lambda\frac{v_0 v}{c^2}}{2\pi\epsilon_0 r'}\hat{r}' = -\frac{\gamma\lambda v_0 v}{2\pi\epsilon_0 c^2 r'}\hat{r}'$$

The test charge will therefore experience a Lorentz force

$$\vec{F}' = q\vec{E}' = -\frac{q\gamma\lambda v_0 v}{2\pi\epsilon_0 c^2 r'}\hat{r}'$$

The force on the moving test charge, measured in the lab frame, is

$$\vec{F} = \frac{\vec{F}'}{\gamma} = -\frac{q\overbrace{\lambda v_0}^{=I} v}{2\pi\epsilon_0 c^2 r'}\hat{r}' = -\frac{qvI}{2\pi\epsilon_0 c^2 r}\hat{r}$$

Since

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

thus, we obtain the force

$$\vec{F} = -\frac{\mu_0 qvI}{2\pi r}\hat{r} = qv\left(\frac{\mu_0 I}{2\pi r}\right)(-\hat{y})$$

Let $\hat{x}\times\hat{z} = -\hat{y}$ and $\vec{v} = v\hat{x}$. The force, in the lab frame, becomes

$$\vec{F} = q\vec{v}\times\frac{\mu_0 I}{2\pi r}\hat{z} = q\vec{v}\times\vec{B}$$

where

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{z}$$

Conclusion: if a charge is stationary, it only produces an electric field, but when viewing the charge from a frame that is moving relative to the charge, a magnetic field is also produced.

NOTE:

An electric field for someone in the stationary frame, might appear as a magnetic field for someone else in the moving frame, because eletric and magnetic fields are not invariant under the Lorentz transformation. That is, two observers may disagree on whether a field "looks" electric or magnetic.